3: Characterizing MNE and Zero-Sum Games

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Characterizing MNE:

Recall our defn. of a mixed Nash Equilibrium:

For n-player games:

$\sigma^* = (\sigma_1^*, \dots, \sigma_n^*)$ is a MNE if, $\forall$ players i, and any alternate mixed strategy $\sigma_i' \in \Sigma_i$,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*)$$

i.e., for each player i, if all players $j \neq i$ are playing σ_j^*, player i maximizes utility by playing σ_i^* (or, σ_i^* is a "best-response" to σ_{-i}^*)

For a 2-player game R, C ($\in \mathbb{R}^{m \times n}$):

x^*, y^* is a MNE if $\forall x' \in \Delta_m$, $y' \in \Delta_n$,

$$x^{*T} R y^* \geq x'^T R y^*$$

and

$$y^{*T} C x^* \geq y'^T C x^*$$

The question we now want to answer is, given a 2-player game R, C, and strategy profile x^*, y^*, can we check in poly-time if (x^*, y^*) is a MNE?

By defn., this is equivalent to verifying if x^* is a best-response to y^*, & vice-versa.

Let's define $BR(y^*) = \arg\max_{x \in \Delta_n} x^T R y^*$

& $BR(x^*) = \arg\max_{y \in \Delta_n} y^T C x^*$

Then by defn., (x^*, y^*) is a MNE if and only if $x^* \in BR(y^*)$, & $y^* \in BR(x^*)$

Okay, so how do we check if $x^* \in BR(y^*)$ & $y^* \in BR(x^*)$?

Example: Recall the penalty shoot-out game:

		G: L	G: R
K	L	$10 \backslash -5$	$-5 \backslash 5$
	R	$-5 \backslash 5$	$10 \backslash -5$

Say G plays $y^* = (1/2, 1/2)$. Then $Ry^* = \begin{bmatrix} 2.5 \\ 2.5 \end{bmatrix}$, and all actions are best-responses, i.e., $BR(y^*) = \Delta_2$.

Say G plays $y^* = (1/3, 2/3)$. Then $Ry^* = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$, and K's best response is to play R w.p. 1, i.e., $BR(y^*) = (0, 1)$.

We show that a mixed action is a best-response iff it is supported on pure best-responses.

For a mixed action $x \in \Delta_m$, define supp$(x) = \{i : x_i > 0\}$ be the set of pure strategies played with positive probability (similarly define supp(y) for $y \in \Delta_n$)

Proposition: For a 2-player game (R, C) and mixed strategies (x^*, y^*), $BR(y^*) = \{x : \text{supp}(x) \subseteq \arg\max_{i \in [m]} (Ry^*)_i\}$

& $BR(x^*) = \{y : \text{supp}(y) \subseteq \arg\max_{j \in [n]} (Cx^*)_j\}$.

Proof: Let $v_R = \max_{i \in [m]} (Ry^*)_i$ be the maximum value obtained by any pure strategy for the row player. Then clearly, for any strategy supported on $\arg\max_{i \in [m]} (Ry^*)_i$, the expected payoff is v_R.

Further, any strategy that puts positive probability on a non-optimal pure strategy (i.e., $x_k > 0$ for $k \notin \arg\max_{i \in [m]} (Ry^*)_i$), the expected payoff is strictly less than v_R.

The proposition follows. ∎

This then gives us an easy way of checking if $x^* \in BR(y^*)$: look at the set of pure strategies w/ max. expected payoff $\arg\max_{i \in [m]} (Ry^*)_i$, & check that supp$(x^*) \subseteq \arg\max_{i \in [m]} (Ry^*)_i$.

Similarly we can check if $y^* \in BR(x^*)$.

If both are best-responses to each other, then (x^*, y^*) is a MNE.

Finding equilibria in zero-sum games:

Let (R, C) be a zero-sum game. Thus, $C = -R^T$.

Consider the column player's perspective. Suppose it plays y. Then the row player's utilities for it's strategies are Ry (this is a column vector).

If the row player chooses best-response to y, it gets $\max_i (Ry)_i$, and hence the column player gets $-\max_i (Ry)_i$.

Since at equilibrium both players best-respond to each other (by defn.), the column player "should" choose y to maximize it's utility when row player best-responds, i.e., choose y to maximize $-\max_i (Ry)_i$

$\equiv$ maximize$_y$ $\min_i (-Ry)_i$

Note that we are not saying that such a y is an equilibrium strategy, in particular why y is a best-response to the row-player's strategy (it may not be!)

But we can find such a y by an LP:

$$P_C: \quad \begin{aligned} \max \quad & z \\ \text{s.t. } \forall i, \ (-Ry)_i &\geq z \\ \sum_j y_j &= 1 \\ y &\geq 0 \end{aligned}$$

Similarly, for the row player, a good strategy would be to choose x which optimizes:

$$P_R: \quad \begin{aligned} \max \quad & w \\ \text{s.t. } \forall j, \ (-Cx)_j &\geq w \\ \sum_i x_i &= 1 \\ x &\geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & w \\ y^*: \quad Cx + w &\leq 0 \\ z'^*: \quad \sum_i x_i &= 1 \\ x &\geq 0 \end{aligned}$$

(rewriting P_R)

Let us write the dual of P_R. This is:

$$\begin{aligned} \min \quad & z' \\ \forall i, \ (C^T y)_i &\geq -z' \\ \sum_j y_j &= 1 \\ y &\geq 0 \end{aligned} \quad \equiv \quad D_R: \ \begin{aligned} \min \quad & -z \\ \forall i, \ (-Ry)_i &\geq z \\ \sum_j y_j &= 1 \\ y &\geq 0 \end{aligned}$$

Note that D_R is nearly the same as P_C, except that the objective value gets negated. I.e., (y^*, z^*) is optimal for D_R iff (y^*, z^*) is optimal for P_C.

Let (x^*, w^*) be optimal for P_R, & (y^*, z^*) be optimal for P_C. Then by strong duality, $-z^* = w^*$.

Theorem: (x^*, y^*) is a NE

Proof: We need to show that for the row player, x^* is a best-response to y^*, i.e., $\forall x$, $x^T R y^* \leq x^{*T} R y^*$.

Consider y^*. We know that if column player plays y^*, and if row-player best-responds, column player gets z^* (negation of D_R). Thus, row-player gets $-z^*$. Thus for any response to y^*, row-player gets at most $-z^*$.

$\forall x \quad x^T R y^* \leq -z^* = w^*$

Now consider x^*. Similar to above, for any strategy y, row player gets at least w^*.

$\forall y \quad x^{*T} R y^* \geq w^*$.

Thus, $x^{*T} R y^* \geq x^T R y^* \ \forall x$, and hence x^* is a best response to y^*.

Similarly we can show that y^* is a best-response to x^*, and hence (x^*, y^*) is a NE ∎

Note:
(i) The proof holds for any optimal soln. x^* to P_R, and any optimal soln. y^* to P_C.
(ii) For any such x^*, y^*, the row-player's utility at equilibrium is w^*. Hence, there are multiple equilibria, but the row-player's payoff (and hence, the column player's payoff) is exactly the same.
The value w^* is called the value of the zero-sum game
(iii) At equilibrium, each player is playing a min-max strategy, or a risk-averse strategy.
In general games, a min-max strategy does not give an equilibrium.

Theorem: Let (x^*, y^*) be a NE of a zero-sum game, & w^*, z^* be payoffs of the two players.
Then (x^*, w^*) is an optimal soln. for P_R, and (y^*, z^*) is an optimal soln for P_C.

Q Prove yourself.

Transforming the payoff matrices

Let R, C be a two-player game, & let x^*, y^* be a MNE for (R, C).

Suppose we add $\lambda \in \mathbb{R}$ to every entry in R. Is (x^*, y^*) still a MNE of this game?

Claim: Given a 2-player game R, C, $j \in [n]$, and $\lambda \in \mathbb{R}$.
Let $R' = R + \lambda \mathbb{1} e_j^T$. Then for any $\hat{x} \in \Delta_m$, $\hat{y} \in \Delta_n$,
$\hat{x} \in BR(\hat{y})$ for matrix R $\iff$ $\hat{x} \in BR(\hat{y})$ for matrix R'

$$R' = R + \begin{array}{c} \quad\quad\quad\quad j \\ \begin{bmatrix} 0 \dots 0 & \lambda & 0 \dots 0 \\ \vdots & \vdots & \vdots \\ 0 \dots 0 & \lambda & 0 \dots 0 \end{bmatrix} \end{array}$$

i.e., adding λ to every elt. in a column in R does not change set of best-responses

Note that this means that (x^*, y^*) is a MNE in R, C iff (x^*, y^*) is a MNE in (R', C)

Proof of claim: Suppose $x^* \in BR(y^*)$ for R.
Then supp$(x^*) \subseteq \arg\max_{i \in [m]} (Ry^*)_i$

We need to show that supp$(x^*) \subseteq \arg\max_{i \in [m]} (R'y^*)_i$

But for any $k \in [m]$, the payoff shifts by exactly λy_j^*:

$$(R'y^*)_k = (Ry^*)_k + \lambda(\mathbb{1} e_j^T y^*)_k$$
$$= (Ry^*)_k + \lambda y_j^*$$

Since each coordinate shifts by the same amount,

$\arg\max_{i \in [m]} (Ry^*)_i = \arg\max_{i \in [m]} (R'y^*)_i$

& hence, supp$(x^*) \subseteq \arg\max_{i \in [m]} (R'y^*)_i$

The other direction follows by symmetry. ∎

Extending the claim, given R, C,

let $R' = R + \begin{bmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \vdots & \vdots & & \vdots \\ \lambda_1 & \lambda_2 & & \lambda_n \end{bmatrix}$

& $C' = C + \begin{bmatrix} \mu_1 & \mu_2 & \dots & \mu_m \\ \vdots & \vdots & & \vdots \\ \mu_1 & \mu_2 & & \mu_m \end{bmatrix}$

Then (x^*, y^*) is MNE for (R, C)
$\iff$ (x^*, y^*) is MNE for (R', C').